Chapter 16: Queueing Theory

: input source: population having hair
calling units: customers wanting haircuts
queue: customers waiting for a barber
service discipline: usually first in, first out.
service mechanism: barbers and equipment.

Part	Customers	Servers
(a)	customers waiting checkout	checkers
(b)	fires	fire fighting units
(c)	cars	toll collectors
(d)	broken bicycles	bicycle repairpersons
(e)	ships to be loaded or unloaded	Longshoremen + equipment
(f)	machines needing operator	operator
(g)	materials to be handled	handling equipment
(h)	calls for plumbers	plumbers
(i)	custom orders	customized process
(j)	typing requests	typists

$\lambda_n = 1/2$ for $n>0$ and $\mu_n = \begin{cases} 1/2 & \text{for } n=1 \\ 1 & \text{for } n\geq 2 \end{cases}$

(a) $P\{\text{next arrival before } 1{:}00\} = 1-e^{-1/2} = .393$

$P\{\text{next arrival between } 1{:}00 \text{ and } 2{:}00\} = (1-e^{-1/2\cdot 2})-(1-e^{-1/2}) = .239$

$P\{\text{next arrival after } 2{:}00\} = e^{-2\cdot 1/2} = .368$

(b) $P\{\text{next arrival between } 1{:}00 \text{ and } 2{:}00 \mid \text{no arrivals between } 12{:}00 \text{ and } 1{:}00\} = 1-e^{-1/2} = .393$

(c) $P\{\text{no arrivals between } 1{:}00 \text{ and } 2{:}00\} = \dfrac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-1/2} = .607$

$P\{\text{one arrival between } 1{:}00 \text{ and } 2{:}00\} = \dfrac{(\lambda t)^1 e^{-\lambda t}}{1!} = \frac{1}{2}\cdot e^{-1/2} = .303$

$P\{\text{two or more arrivals between } 1{:}00 \text{ and } 2{:}00\} = 1-e^{-1/2}-\frac{1}{2}e^{-1/2} = .090$

(d) $P\{\text{none served by } 2{:}00\} = e^{-1} = .368$

$P\{\text{none served by } 1{:}10\} = e^{-1(1/6)} = .846$

$P\{\text{none served by } 1{:}01\} = e^{-1(1/60)} = .983$

$\lambda_n = 2$ for $n \geq 0 \Rightarrow P\{n \text{ arrivals in an hour}\} = \dfrac{2^n e^{-2}}{n!}$

(a) $P\{0 \text{ arrivals in an hour}\} = e^{-2} = .135$

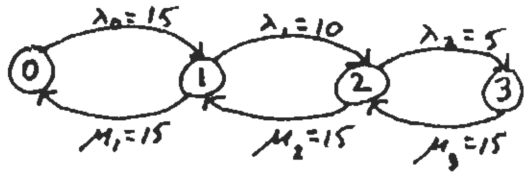
4 (continued)

b) $P\{2 \text{ arrivals in an hour}\} = \frac{2^2 e^{-2}}{2!} = 2e^{-2} = .270$

c) $P\{5 \text{ or more arrivals in an hour}\} = 1 - \sum_{n=0}^{4} P\{n \text{ arrivals in an hour}\}$

$= 1 - e^{-2} - 2e^{-2} - 2e^{-2} - (4/3)\cdot e^{-2} - (2/3)e^{-2}$
$= 1 - 7e^{-2} = .0527$

5. (a)



(b) $15P_1 = 15P_0$

$15P_0 + 15P_2 = 25P_1$

$10P_1 + 15P_3 = 20P_2$

$5P_2 = 15P_3$

(c) Solving the above system we obtain:
$(P_0, P_1, P_2, P_3) = (9/26, 9/26, 3/13, 1/13) = (.35, .35, .23, .07)$
Now using the general solution for the birth and death process

$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{2}{3} P_0$

$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{2}{9} P_0$

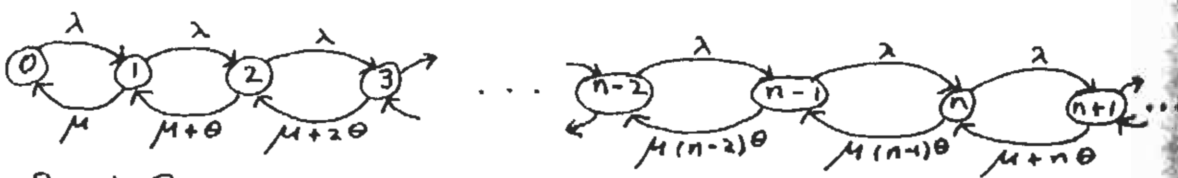
and $P_0 + P_1 + P_2 + P_3 = 1 \Rightarrow (P_0, P_1, P_2, P_3) = (9/26, 9/26, 3/13, 1/13)$ as previously found.

(d) $L = \sum_{n=0}^{3} nP_n = 27/26 = 1.04$

$\bar{\lambda} = \sum_{n=0}^{2} \lambda_n P_n = 255/26 = 9.81$

So $W = L/\bar{\lambda} = 27/255 = .11$ hours

6. (a)



(b) $\mu P_1 = \lambda P_0$

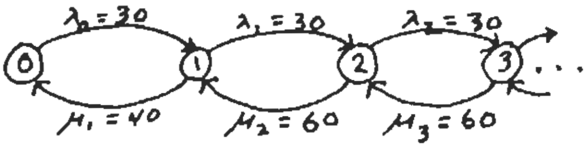
$\lambda P_0 + (\mu + \theta)P_2 = (\mu + \lambda)P_1$

$\vdots$

$\lambda P_{n-1} + (\mu + n\theta)P_{n+1} = (\lambda + \mu + (n-1)\theta)P_n$

$\vdots$

1 (a)



(b) $P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu_1 \mu_2^{n-1}}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1} \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu_2}\right)^{n-1}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1}\left(\frac{1}{1-\frac{\lambda}{\mu_2}}\right)\right]^{-1}$

$= \left[1 + \frac{3}{4}\left(\frac{1}{1-1/2}\right)\right]^{-1} = 2/5 = .4$

$P_n = P_0 \frac{\lambda^n}{\mu_1 \mu_2^{n-1}} = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)^n$ for $n \geq 1$

c) $L = \sum_{n=0}^{\infty} n P_n = \frac{3}{5} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{5} \cdot \frac{1}{2} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{(1-1/2)^2} = \frac{6}{5}$

$L_q = L - (1 - P_0) = 6/5 - (1 - 2/5) = 3/5$

$W = L/\lambda = 1/25$

$W_q = L_q/\lambda = 1/50$

8. (a) $L_q = W_q = 0$

(b)

(c) $P_n = \frac{\lambda^n}{n!\mu^n} P_0 = \frac{\rho^n}{n!} P_0$

(d) $P_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{\rho^n}{n!}} = \frac{1}{e^{\rho}} = e^{-\rho}$ notice that ρ need not be ≤ 1 in this case

(e) $W = 1/\mu$, $L = \lambda W = \lambda/\mu = \rho$

9 (a) $\lambda P_0 = \mu P_1$ . . . . (0)

$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_n$ . . . (1)

$\vdots$

$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$ . . . (n)

The solution given in Sec. 16.6 is:

$P_n = (1-\rho)\rho^n$ for $n = 0, 1, 2 \ldots$

Verifying that the above satisfy the balance equations:

equation (0): $\lambda \cdot (1-\rho) = \mu (1-\rho) \cdot \rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \frac{\lambda}{\mu}$ OK

equation (n): $\lambda (1-\rho)\rho^{n-1} + \mu (1-\rho)\rho^{n+1} = (\lambda + \mu)(1-\rho)\rho^{n}$

$\Leftrightarrow \lambda + \mu\rho^2 = (\lambda + \mu) \cdot \rho$

$\lambda + \frac{\lambda^2}{\mu} \qquad \frac{\lambda^2}{\mu} + \lambda$ OK

9. (continued)

(b) $\lambda P_0 = \mu P_1$

$\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 16.6 is:

$$P_n = \left(\frac{1-\rho}{1-\rho^3}\right)\rho^n \quad \text{for } n = 0,1,2$$

Verifying:

- $\lambda \cdot \frac{1-\rho}{1-\rho^3} = \mu \cdot \frac{1-\rho}{1-\rho^3}\cdot\rho \Leftrightarrow \lambda = \mu\cdot\rho = \mu\cdot\frac{\lambda}{\mu}$ . . . <u>OK</u>
- $\lambda\cdot\frac{1-\rho}{1-\rho^3} + \mu\frac{1-\rho}{1-\rho^3}\cdot\rho^2 = (\lambda+\mu)\frac{1-\rho}{1-\rho^3}\rho \Leftrightarrow \underbrace{\lambda+\mu\cdot\rho^2}_{=\,\lambda+\frac{\lambda^2}{\mu}} = \underbrace{(\lambda+\mu)\cdot\rho}_{=\,\frac{\lambda^2}{\mu}+\lambda}$ . . . <u>OK</u>
- $\lambda\cdot\frac{1-\rho}{1-\rho^3}\cdot\rho = \mu\frac{1-\rho}{1-\rho^3}\cdot\rho^2 \Leftrightarrow \underbrace{\lambda\cdot\rho}_{=\,\frac{\lambda^2}{\mu}} = \underbrace{\mu\rho^2}_{=\,\frac{\lambda^2}{\mu}}$ . . . <u>OK</u>

(c) $2\lambda P_0 = \mu P_1$

$2\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$

$2 P_1 = \mu P_2$

The solution given in Sec. 16.6 is:

$$P_0 = \left[\sum_{n=0}^{2}\frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$$

$$P_n = \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad \text{for } n = 1,2$$

Verifying:

- $2\lambda/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2) = \mu\cdot 2(\frac{\lambda}{\mu})/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2)$

  $\Leftrightarrow 2\lambda = \mu\cdot 2\cdot\frac{\lambda}{\mu}$ . . . <u>OK</u>
- $2\lambda/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2)+\mu\cdot 2(\frac{\lambda}{\mu})^2/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2) =$

  $= (\lambda+\mu)2(\lambda/\mu)/(1+2(\lambda/\mu)+2(\lambda/\mu)^2) \Leftrightarrow \underbrace{2\lambda+2\mu(\frac{\lambda}{\mu})^2}_{=\,2(\lambda+\frac{\lambda^2}{\mu})} = \underbrace{2(\lambda+\mu)(\frac{\lambda}{\mu})}_{=\,2(\frac{\lambda^2}{\mu}+\lambda)}$ . . . <u>OK</u>
- $\lambda\cdot 2(\frac{\lambda}{\mu})/(1+2(\lambda/\mu)+2(\lambda/\mu)^2) = \mu\cdot 2\cdot(\lambda/\mu)^2/(1+2(\lambda/\mu)+2(\lambda/\mu)^2)$

  $\Leftrightarrow 2\frac{\lambda^2}{\mu} = 2\frac{\lambda^2}{\mu}$ . . . <u>OK</u>

10.

(a)

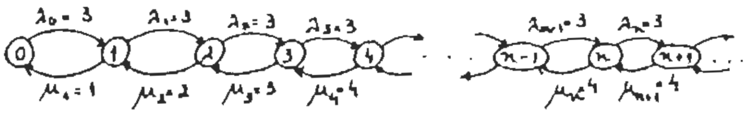
λ	L	L_q	W	W_q	$P\{\mathcal{W}>5\}$
.5	1	.50	2	1	.082
.9	9	8.10	10	9	.607
.99	99	98.01	100	99	.951

(b)

λ	λ/μ	ρ	P_0	L	L_q	W	W_q	$P\{\mathcal{W}>5\}$
.5	1	.5	.3333	1.333	.333	2.667	.667	.150
.9	1.8	.9	.0526	9.474	7.674	10.526	8.526	.641
.99	1.98	.99	.0050	99.497	97.517	100.503	98.503	.956

11.

(a)



(b) $P_0 = \left[\sum_{n=0}^{3} \frac{3^n}{n!} + \frac{3^4}{4!}\cdot\frac{1}{1-3/4}\right]^{-1} = \frac{2}{53} = .038$

$P_1 = \left(\frac{\lambda}{\mu}\right)P_0 = 3\cdot\frac{2}{53} = \frac{6}{53} = .113$

$P_2 = \frac{(\lambda/\mu)^2}{2!}P_0 = \frac{9}{2}\cdot\frac{2}{53} = \frac{9}{53} = .170$

$P_3 = \frac{(\lambda/\mu)^3}{3!}P_0 = \frac{27}{6}\cdot\frac{2}{53} = \frac{9}{53} = .170$

$P_4 = \frac{(\lambda/\mu)^4}{4!}P_0 = \frac{81}{24}\cdot\frac{2}{53} = \frac{27}{212} = .127$

$P_n = \frac{(\lambda/\mu)^n}{4!\,4^{(n-4)}}P_0 = \frac{3^n}{4\cdot3\cdot2\cdot4^{n-4}}\cdot\frac{2}{53} = \frac{3^{n-1}}{4^{n-3}}\cdot\frac{1}{53} = \frac{9}{53}\left(\frac{3}{4}\right)^{n-3}$ for $n=4,5,\ldots$

(c) $L_q = \sum_{n=4}^{\infty}(n-4)\cdot P_n = \sum_{n=4}^{\infty}(n-4)\cdot\frac{9}{53}\left(\frac{3}{4}\right)^{n-3} =$

$= \frac{9}{53}\cdot\left(\frac{3}{4}\right)^2\sum_{n=4}^{\infty}(n-4)\left(\frac{3}{4}\right)^{n-5} = \frac{9}{53}\cdot\left(\frac{3}{4}\right)^2\cdot\frac{1}{(1-3/4)^2} = \frac{81}{53} = 1.528$

$W_q = L_q/\lambda = 27/53 = .509$

$W = W_q + 1/\mu = (27/53) + 1 = 80/53 = 1.509$

$L = \lambda W = 240/53 = 4.528$

12. $\lambda = 2$, $\mu = 4$, $s = 1$, $\rho = 1/2$

For M/M/1 queue, $P_0 = 1 - \lambda/\mu = 1/2$ and $P_n = (1-\rho)\rho^n = (1/2)^{n+1}$

desired proportion of time $= \sum_{i=0}^{4} P_i = 31/32$

13. $\lambda = 3$, $\mu = 4$, $s = 1$, $\rho = 3/4$

The system without the storage restriction is a M/M/1 queue. If n square feet of floor space were available for waiting, the proportion of time this would be sufficient is $\sum_{i=0}^{n+1} P_i$. Thus we want to find n_ℓ such that $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell$ for $\ell = 1,2,3$, where $q_1 = .5, q_2 = .9, q_3 = .99$

Now $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell \iff \sum_{i=0}^{n_\ell+1}(1-\rho)\rho^i \geq q_\ell \iff (1-\rho)\,\frac{(1-\rho^{n_\ell+2})}{(1-\rho)} \geq q_\ell \iff$

$\iff 1-\rho^{n_\ell+2} \geq q_\ell \iff \rho^{n_\ell+2} \leq 1-q_\ell \iff (n_\ell+2)\ln\rho \leq \ln(1-q_\ell)$

$\iff (n_\ell+2) \geq \frac{\ln(1-q_\ell)}{\ln\rho} \iff n_\ell \geq \frac{\ln(1-q_\ell)}{\ln\rho} - 2$

13 (continued)

part	q_l	$\frac{\ln(1-q_l)}{\ln \rho} - 2$	floor space required
(a)	.50	.409	1
(b)	.90	6.004	7
(c)	.99	14.008	15

14. Let $P\{\mathcal{W} \le t\} = G(t)$ and let $\frac{d\,G(t)}{dt} = g(t)$

Then $P\{\mathcal{W} > t\} = 1 - P(t)$.

So $[1 - G(t)] = \sum_{n=0}^{\infty} P_n\, P\{S_{n+1} > t\} =$

$$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\int_t^{\infty} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!}\, dx\right] =$$

$$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^{n+1} x^n e^{-\mu x}}{n!}\, dx\right]$$

Differentiating both sides, we have:

$$g(t) = \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^{n+1} t^n e^{-\mu t}}{n!}\right]$$

$$= (1-\rho)\,\mu\, e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$$

$$= (1-\rho)\,\mu\, e^{-\mu t} e^{\lambda t}$$

$$= \mu(1-\rho)\, e^{-\mu(1-\rho)t}$$

Hence, by integration, $P\{\mathcal{W} > t\} = 1 - \int_0^t g(x)\,dx = e^{-\mu(1-\rho)t}$

15. (a) Let $P\{\mathcal{W}_q \le t\} = G(t)$ and let $\frac{d\,G(t)}{dt} = g(t)$

Then $P\{\mathcal{W}_q > t\} = 1 - G(t)$

So $[1 - G(t)] = \sum_{n=1}^{\infty} P_n\, P\{S_n > t\} = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!}\, dx\right]$

Differentiating both sides gives:

$$g(t) = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!}\right]$$

$$= (1-\rho)\,\lambda\, e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= (1-\rho)\,\lambda\, e^{-\mu t} e^{\lambda t}$$

$$= \left(\frac{\lambda}{\mu}\right)(\mu-\lambda)\, e^{-(\mu-\lambda)t}$$

Then $W_q = \left(\frac{\lambda}{\mu}\right) \int_0^{\infty} t\,(\mu-\lambda) e^{-(\mu-\lambda)t}\, dt = \frac{\lambda}{\mu(\mu-\lambda)}$

(b) Let $P\{\mathcal{W}_q < t\} = G(t)$ and let $\frac{d\,G(t)}{dt} = g(t)$

Then $P\{\mathcal{W}_q > t\} = 1 - G(t)$

So $[1 - G(t)] = \sum_{n=s}^{\infty} P_n\, P\{S_{n-s+1} > t\}$

$$= \sum_{n=s}^{\infty} P_n \left[1 - \int_0^t \frac{(s\mu)^{n-s+1} x^{n-s} e^{-(s\mu)x}}{(n-s)!}\, dx\right]$$

Now $P_n = \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$ for $n \ge s$

So differentiating both sides gives

$$g(t) = \sum_{n=s}^{\infty} \left[\frac{(\lambda/\mu)^n P_0}{s!\, s^{n-s}}\right] \left[\frac{(s\mu)^{n-s+1} t^{n-s} e^{-(s\mu)t}}{(n-s)!}\right]$$

15 (continued)

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} \sum_{n=s}^{\infty} \frac{(\lambda t)^{n-s}}{(n-s)!}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} e^{\lambda t}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-(s\mu)(1-\rho)t}$$

So $$W_q = \frac{P_0 (\lambda/\mu)^s}{s!} \int_0^{\infty} t (s\mu) e^{-(s\mu)(1-\rho)t} dt$$

$$= \frac{P_0 (\lambda/\mu)^s}{s!(1-\rho)} \int_0^{\infty} t (s\mu)(1-\rho) e^{-(s\mu)(1-\rho)t} dt$$

$$= \frac{P_0 (\lambda/\mu)^s}{s!(1-\rho)^2 (s\mu)}$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!(1-\rho)^2 \lambda}$$

$$= L_q / \lambda$$

16. W and W_q represent the waiting times of arriving customers who enter the system. The probability that such a customer finds n customers already there is:

$$P\{n \text{ customers in system} \mid \text{system not full}\} = \begin{cases} \frac{P_n}{1-P_K} & 0 \le n \le K-1 \\ 0 & n = K \end{cases}$$

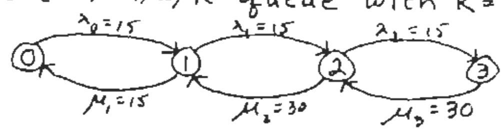
And so:

(a) $$P\{W > t\} = \frac{1}{1-P_K} \sum_{n=0}^{K-1} P_n P\{S_{n+1} > t\}$$

(b) $$P\{W_q > t\} = \frac{1}{1-P_K} \sum_{n=0}^{K-1} P_n P\{S_n > t\}$$

17. This is a M/M/2/K queue with $K=3$, $\lambda=15$, $\mu=15$.

(a)



(b) $$P_0 = \left[1 + \sum_{n=1}^{2} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^2}{2!} \sum_{n=3}^{3} \left(\frac{\lambda}{2\mu}\right)^{n-2}\right]^{-1}$$

$$= \left[1 + 1 + \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2}\right)\right]^{-1} = \frac{4}{11} = .364$$

$P_1 = (\lambda/\mu) P_0 = 4/11 = .364$

$P_2 = \frac{(\lambda/\mu)^2}{2!} P_0 = \frac{1}{2} \cdot \frac{4}{11} = \frac{2}{11} = .182$

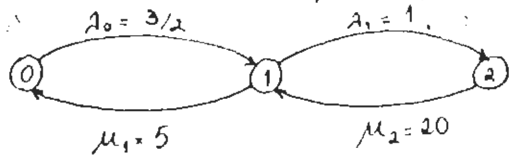
$P_3 = \frac{(\lambda/\mu)^3}{2!\,2^{3-2}} P_0 = \frac{1}{4} \cdot \frac{4}{11} = \frac{1}{11} = .091$

So $P\{(i)\} = P_0 + P_1 = 8/11 = .727$

$P\{(ii)\} = P_2 = 2/11 = .182$

$P\{(iii)\} = P_3 = 1/11 = .091$

18. (a) State n = # machines broken down
Since the 3rd machine is shut off when the 2nd machine breaks, $n = 0, 1$ or 2.



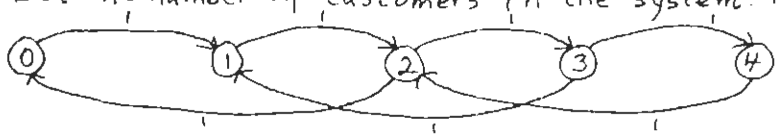
(b) The balance equations are:

$$5P_1 = (3/2)P_0$$
$$(3/2)P_0 + 20P_2 = 6P_1$$
$$P_1 = 20P_2$$

Solving the balance equations we obtain
$$(P_0, P_1, P_2) = (200/263,\ 60/263,\ 3/263) \approx (.760, .228, .011)$$

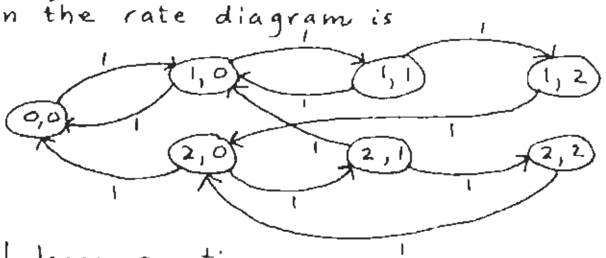
(c) $E[\#\text{ operators available}] = 2P_0 + 2P_1 + 0P_2 =$
$= 520/263 = 1.977$

19. (a) Let n = number of customers in the system. Then the rate diagram is:



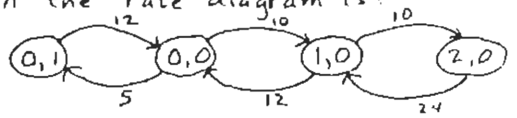
The balance equations are:
$$P_0 = P_2$$
$$P_1 = P_0 + P_3$$
$$2P_2 = P_1 + P_4$$
$$2P_3 = P_2$$
$$P_4 = P_3$$

(b) The state space has to be more complex in this case because you need to know how many customers are being worked on by the server.
Let the state be (s, q) where
s = number of customers being served
q = number of customers in the queue
Then the rate diagram is



The balance equations are:
$$P_{00} = P_{10} + P_{20}$$
$$2P_{10} = P_{00} + P_{11} + P_{21}$$
$$2P_{11} = P_{10}$$
$$P_{12} = P_{11}$$
$$2P_{20} = P_{12} + P_{22}$$
$$2P_{21} = P_{20}$$
$$P_{22} = P_{21}$$

20.(a) Let the state be (n_1, n_2) where
n_1 = number of type 1 customers in the system
n_2 = number of type 2 customers in the system
Then the rate diagram is:



(b) The balance equations are:

$$12 P_{01} = 5 P_{00}$$
$$15 P_{00} = 12 (P_{01} + P_{10})$$
$$22 P_{10} = 10 P_{00} + 24 P_{20}$$
$$24 P_{20} = 10 P_{10}$$
$$(P_{00} + P_{10} + P_{01} + P_{20} = 1)$$
$$\Rightarrow P_{00} = \frac{72}{187},\ P_{10} = \frac{60}{187},\ P_{01} = \frac{30}{187},\ P_{20} = \frac{25}{187}$$

(c) Type 1 customers are blocked when the system is in state (2,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{01} = 55/187$.
Type 2 customers are blocked when the system is in state (2,0),(1,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{10} + P_{01} = 115/187$.

21. This is a M/M/1/K queue with $K = 1, 3$ and 5, respectively. Also, $\lambda = 1/4$ and $\mu = 1/3$ so that $\rho = 3/4$
The fraction of customers lost $= P_K = \frac{(1-\rho)}{(1-\rho^{K+1})} \rho^K$

(a) zero spaces. $P_1 = \frac{(1-3/4)}{(1-(3/4)^2)} (3/4) = \frac{3}{7} = .429$

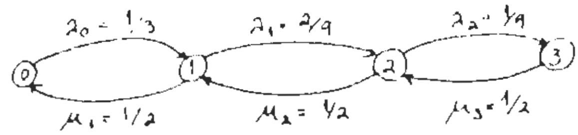
(b) two spaces: $P_3 = \frac{(1-3/4)}{(1-(3/4)^4)} (3/4)^3 = \frac{27}{175} = .154$

(c) four spaces: $P_5 = \frac{(1-3/4)}{(1-(3/4)^6)} (3/4)^5 = \frac{243}{3367} = .072$

22. M/M/s/K model

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n =$$
$$= \sum_{n=s}^{K} (n-s) \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$$
$$= \frac{P_0 (\lambda/\mu)^{s+1}}{s!\ s} \sum_{n=s}^{K} (n-s) \left(\frac{\lambda}{s\mu}\right)^{n-s-1} =$$
$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} j \rho^{j-1} =$$
$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} \frac{d(\rho^j)}{d\rho} =$$
$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \frac{d}{d\rho} \left[\sum_{j=0}^{K-s} \rho^j\right] =$$
$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \frac{d}{d\rho} \left(\frac{1-\rho^{K-s+1}}{1-\rho}\right) =$$
$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \left[\frac{1-\rho^{K-s} - (K-s)\rho^{K-s}(1-\rho)}{(1-\rho)^2}\right]$$

23. (a)



Using the steady state solution of the birth and death process:

$P_0 = \left[1 + \frac{2}{3} + \frac{8}{27} + \frac{16}{243}\right]^{-1} = \frac{243}{493} = .493$

$P_1 = \frac{2}{3} \cdot P_0 = \frac{162}{493} = .329$

$P_2 = \frac{8}{27} P_0 = \frac{72}{493} = .146$

$P_3 = \frac{16}{243} P_0 = \frac{16}{493} = .032$

E[# not running] = $354/493 = .718$

(b) (i) This is a M/M/1 queue with $\lambda = 1/3$ and $\mu = 1/2$

So $P_0 = 1 - \rho = 1/3 = .333$

$P_1 = (1-\rho)\rho = 2/9 = .222$

$P_2 = (1-\rho)\rho^2 = 4/27 = .148$

$P_3 = (1-\rho)\rho^3 = 8/81 = .099$

$L = \lambda/(\mu - \lambda) = \rho/(1-\rho) = 2$

(ii) This is a M/M/1/K queue with $K=3, \lambda = 1/3, \mu = 1/2$

Thus, $P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1 - 2/3}{1-(2/3)^4} = \frac{27}{65} = .415$

$P_1 = \rho P_0 = \frac{2}{3} \cdot \frac{27}{65} = \frac{18}{65} = .277$

$P_2 = \rho^2 P_0 = \frac{4}{9} \cdot \frac{27}{65} = \frac{12}{65} = .185$

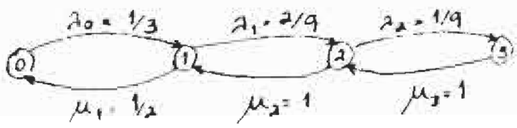
$P_3 = \rho^3 P_0 = \frac{8}{27} \cdot \frac{27}{65} = \frac{8}{65} = .123$

$L = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{66}{65} = 1.015$

Summarizing,

Part	P_0	P_1	P_2	P_3	L
(a)	.493	.329	.146	.032	.718
(b)(i)	.333	.222	.148	.099	2.000
(ii)	.415	.277	.185	.123	1.015

(c)



$P_0 = \left[1 + \frac{2}{3} + \frac{4}{27} + \frac{4}{243}\right]^{-1} = \frac{243}{445} = .546$

$P_1 = (2/3) P_0 = 162/445 = .364$

$P_2 = (4/27) P_0 = 36/445 = .081$

$P_3 = (4/243) P_0 = 4/445 = .009$

E[# not running] = $246/445 = .553$

24. (a) The model for each operator with the machines assigned to him is a M/M/1 queue with limited source. Also, $\lambda = 1/150$, $\mu = 1/15$, and so $\rho = 1/10$

Let N be the number of machines assigned to an operator. Then

$$L = N - (\mu/\lambda)(1-P_0) = N - 10(1-P_0)$$

where $P_0 = \left[\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{1}{10}\right)^n\right]^{-1}$

It is desired that $L \leq .11N$

N	P_0	L	Is $L \leq .11N$?
1	10/11	.091	yes
2	50/61	.197	yes
3	500/683	.321	yes
4	1250/1933	.467	no

So the operator can be assigned at most 3 machines

(b) E[fraction of time operator will be busy] = $1 - P_0$

$= 183/683 = .268$

25. $\mu_1 = 1/8$

(a) $\mu_4 = 1/4 = 4^c(1/8) \Rightarrow 2 = 4^c \Rightarrow c = 1/2$

(b) $\mu_4 = 1/5 = 4^c(1/8) \Rightarrow 8/5 = 4^c \Rightarrow c = \frac{\ln(8/5)}{\ln 4} = .339$

26. Using the L's of figure 16.10 and those calculated in problem 9,

$\frac{\lambda_0}{s\mu_1}$	c	.20	.40	.60
.50	s = 1	.7800	.6500	.6000
	s = 2	.9000	.8475	.7875
.90	s = 1	.2444	.1667	.1333
	s = 2	.3464	.2624	.2257
.99	s = 1	.0288	.0177	.0139
	s = 2	.0454	.0303	.252

27. (a) (i) exponential: $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) constant: $W_q = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

(iii) Erlang: $\sigma = \frac{1}{2}\left(0 + \frac{1}{\mu}\right) = \frac{1}{2\mu} \Rightarrow \sigma^2 = \frac{1}{4\mu^2} \Rightarrow K = 4$

$$W_q = \frac{1+4}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{8}\,\frac{\lambda}{\mu(\mu-\lambda)}$$

So $W_q^{exp} = 2\,W_q^{c} = (8/5)\,W_q^{Erlang}$

27 (continued)

(b) Let $B = 1$, $(1/2)$ and $(5/8)$ when the distribution is exponential, constant or Erlang, respectively

Now $\lambda^{(2)} = 2\lambda^{(1)}$ and $\mu^{(2)} = 2\mu^{(1)}$

$$W_q^{(2)} = B\left[\frac{2\lambda^{(1)}}{2\mu^{(1)}(2\mu^{(1)} - 2\lambda^{(1)})}\right] = \frac{W_q^{(1)}}{2}$$

$$L_q^{(2)} = \lambda^{(2)} W_q^{(2)} = 2\lambda^{(1)} W_q^{(1)}/2 = \lambda^{(1)} W_q^{(1)} = L_q^{(1)}$$

So the waiting time is cut in half while the queue lenght is unchanged

28 (a) Poisson Input

Erlang service with $\mu = 1/12$ and $k = 4$

(b) Poisson Input

General service with mean $3 + 9 = 12$ and

variance $\frac{1}{(k\mu_1^2)} + \frac{1}{(k\mu_2^2)} = 30$

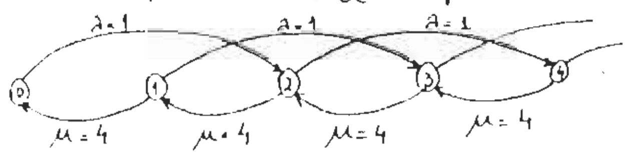
29. Under the current policy, the queueing system has Poisson input with $\lambda = 1$ and exponential service with $\mu = 2$ Under the proposal, the queueing system would have exponential Poisson input with $\lambda = 1/4$ and Erlang service with $\mu = 1/2$ and $K = 4$ On the basis of the information provided, there is no apparent difference in the service cost of the two alternatives. Therefore, any differences in cost must be due to differences in waiting costs of the planes. Let C denote the cost per day of an idle airplane Then, as explained in Section 11.3 of Chapter 17, CL is the expected waiting costs per day for both alternatives. Thus a comparison of the two alternatives reduces to a comparison of their respective values for L.

For the current policy. $L = \frac{\lambda}{\mu - \lambda} = \frac{1}{2-1} = 1$

For the proposal, $L = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu} = \frac{13}{16}$

Hence the proposal should be adopted

30. (a)



$\mu P_1 = \lambda P_0$

$\mu P_2 = (\lambda + \mu) P_1$

$\lambda P_0 + \mu P_3 = (\lambda + \mu) P_2$

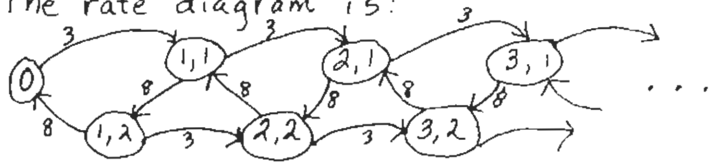
$\lambda P_{n-2} + \mu P_{n+1} = (\lambda + \mu) P_n$

(b) Poisson input with $\lambda = 1$ and Erlang Service with $\mu = 4/2 = 2$ and $k = 2$.

$$W = \left(\frac{1+k}{2k}\right)\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{7}{8}$$

31. (a) Let the state be (n, s) where
$n = \#$ airplanes at the base
$s =$ stage of service of the airplane being overhauled

(b) The rate diagram is:



(c) $L_q = \frac{27}{16}$ airplanes, $L = \frac{39}{16}$ airplanes, $W_q = \frac{9}{16}$ weeks, $W = \frac{13}{16}$ weeks

32. (a) $L = \begin{cases} L_q & \text{when no one is in the system} \\ L_q + 1 & \text{otherwise} \end{cases}$

So $L = P_0 L_q + (1 - P_0)(L_q + 1) = L_q + (1 - P_0)$

(b) $L = \lambda W = \lambda(W_q + 1/\mu) = \lambda W_q + \lambda/\mu = L_q + \rho$

(c) $L = L_q + \rho = L_q + (1 - P_0)$ from (a) and (b). So $\rho = 1 - P_0$.

33. $$L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{\infty} n P_n = \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{\infty} (n-s) P_n + \sum_{n=s}^{\infty} s P_n =$$
$$= \sum_{n=0}^{s-1} n P_n + L_q + s \sum_{n=s}^{\infty} P_n =$$
$$= \sum_{n=0}^{s-1} n P_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

34. For the current arrangement, $\lambda = 24$ and $\mu = 30 \Rightarrow \rho = .8$
For the proposal, $\lambda = 48$, $\mu = 30$ and $s = 2 \Rightarrow \rho = .8$

Model	Current			Proposal	
	L at each crib	Total L	$W = L/\lambda$	L	$W = L/\lambda$
Figure 16.7	4.0	8.0	0.167	4.444	0.093
Figure 16.11	2.4	4.8	0.098	3.1	0.064
Figure 16.13	3.2	6.4	0.133	3.7	0.078
Figure 16.14	2.2	4.4	0.091	2.8	0.058

35.

	W_{q1}	L_{q1}	W_1	L_1	W_{q2}	L_{q2}	W_2	L_2
$s=1$, $\mu=10$	.133	.533	.233	.933	.667	2.667	.767	3.067
$s=2$, $\mu=5$	.119	.474	.319	1.274	.593	2.370	.793	3.170

If W_1 is the primary concern, one should choose the first alternative (one fast server). On the other hand, if W_{q1} is the primary concern, one should choose the second alternative (two slow servers)

36 $\lambda = 8$, $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\mu = 10$

(a) First come, first served: $W = \frac{1}{\mu - \lambda} = \frac{1}{2}$ days

(b) Nonpreemptive

$A = \frac{\mu^2}{\lambda} = \frac{25}{2}$

$B_1 = 1 - (\lambda_1/\mu) = 4/5$

$B_2 = 1 - (\lambda_1 + \lambda_2)/\mu = 2/5$

$B_3 = 1 - \lambda/\mu = 1/5$

36 (b) (continued)

so $W_1 = \frac{1}{AB_1} + \frac{1}{\mu} = \frac{1}{5} = .20$ days

$W_2 = \frac{1}{AB_1B_2} + \frac{1}{\mu} = \frac{7}{20} = .35$ days

$W_3 = \frac{1}{AB_2B_3} + \frac{1}{\mu} = \frac{11}{10} = 1.1$ days

(c) Preemptive:

$W_1 = \frac{1/\mu}{B_1} = \frac{1}{8} = .125$ days

$W_2 = \frac{1/\mu}{B_1B_2} = \frac{5}{16} = .3125$ days

$W_3 = \frac{1/\mu}{B_2B_3} = \frac{5}{4} = 1.25$ days

37. $\lambda_1 = 0.1,\ \lambda_2 = 0.4,\ \lambda_3 = 1.5,\quad \lambda = \sum_{i=1}^{3} \lambda_i = 2\ ,\ \mu = 3$

	Preemptive Priorities		Nonpreemptive Priorities	
	$s=1$	$s=2$	$s=1$	$s=2$
A	...	...	4.5	36
B_1	.967	...	.967	.983
B_2	.833	...	.833	.917
B_3	.333	...	.333	.667
$W_1 - \frac{1}{\mu}$	.011	.00009	.230	.028
$W_2 - \frac{1}{\mu}$	.080	.00289	.276	.031
$W_3 - \frac{1}{\mu}$	.867	.05493	.800	.045

38. For the group of ten machines, the input is a Poisson process with $\lambda = 70$ per hour and the output is exponential with $\mu = 10$ per hour. So $100 = s\mu > \lambda = 70$. Thus, by the Equivalence Property of Section 16.9, the steady state output of the machines (which is the input to the inspector) is also a Poisson process with $\lambda = 70$. So the input of jobs to the inspector is a Poisson process with $\lambda = 70$ per hour and his inspection time is Erlang with parameters $\mu = 80$ and $k = 25$. Hence,

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) = \left(\frac{26}{50}\right)\left[\frac{4900}{800}\right] = 3.185$$

Thus, under the status quo, 3.185 jobs await inspection in steady state.

(a) Proposal 1 would decrease μ for the machine from $\mu = 10$ to $\mu = 8.571$. However $100 = s\mu > \lambda = 70$, so the Equivalence Property again implies that the input of jobs to the inspector is Poisson with $\lambda = 70$. Hence, Proposal 1 would cause no change; that is, $L_q = 3.185$ would still hold. This could be explained by saying that the power decrease in the machines was not sufficiently great to cause a decrease in the output of the machines, which is input to the inspector.

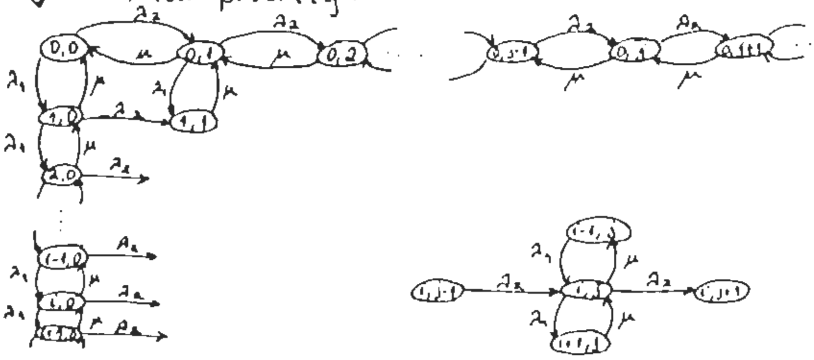
38. (continued)

(b) For Proposal 2, the input to the inspector is Poisson with $\lambda = 70$ as before, but the inspection time is now Erlang with $\mu = 83\,1/3$ and $K = 2$. So

$$L_q = \left(\frac{3}{4}\right)\left(\frac{4900}{250/3\,(250/3 - 70)}\right) = 3.308$$

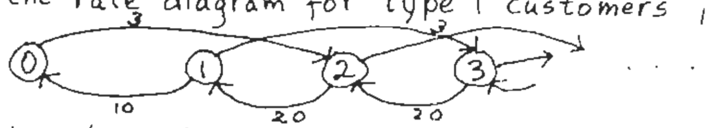
So the expected queue lenght barely changes. This is explained by the variability due to the inspector's lack of experience.

39. Let state (i,j) denote i jobs of high priority and j jobs of low priority.



State	Rate in = Rate out
$(0,0)$	$\mu(P_{1,0} + P_{0,1}) = (\lambda_1 + \lambda_2) P_{0,0}$
$(i,0)$ for $i \geq 1$	$\mu P_{i+1,0} + \lambda_1 P_{i-1,0} = (\mu + \lambda_1 + \lambda_2) P_{i,0}$
$(0,j)$ for $j \geq 1$	$\mu(P_{1,j} + P_{0,j+1}) + \lambda_2 P_{0,j-1} \quad (\mu + \lambda_1 + \lambda_2) P_{0,j}$
(i,j) for $i,j \geq 1$	$\mu P_{i+1,j} + \lambda_1 P_{i-1,j} + \lambda_2 P_{i,j-1} = (\mu + \lambda_1 + \lambda_2) P_{i,j}$

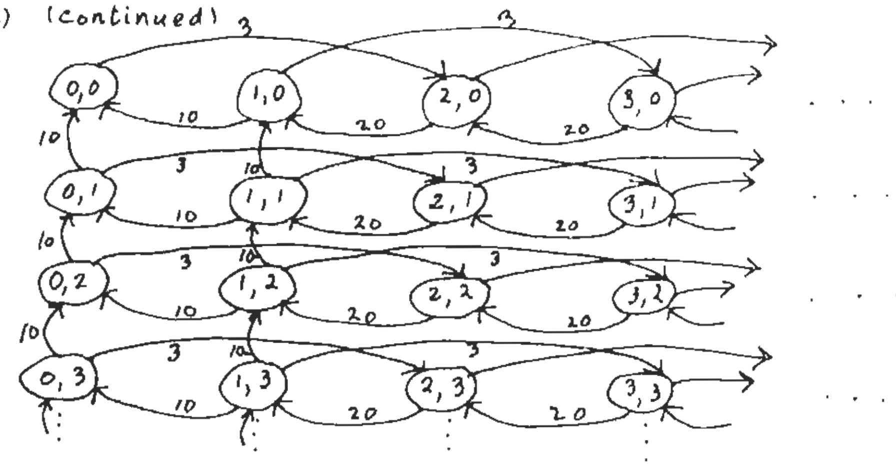
40. (a) Let the state be n_1 = number of type 1 customers in the system. Then the rate diagram for type 1 customers is:



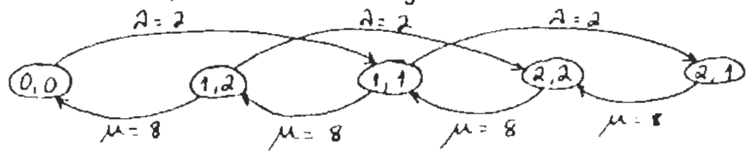
(b) Let the state be n = number of customers in the system. Then the rate diagram for the total number of customers is:

(c) Let the state be (n_1, n_2) where
n_1 = number of type 1 customers in the system
n_2 = number of type 2 customers in the system
Then the rate diagram is:

40.(c) (continued)



41. Let state (i,j) denote i calling units in the system, with the calling unit being served at the j^{th} stage of his Service. Then the state space is: $\{(0,0),(1,2),(1,1),(2,2),(2,1)\}$. The rate diagram is.



Note this analysis is possible because an Erlang distribution with $1/\mu = 1/4$ and $k=2$ is equivalent to the sum of two independent exponentials with parameter $1/\mu = 1/8$.

Hence, the steady state equations are:

$8P_{1,2} = 2P_{0,0}$

$8P_{1,1} = 10P_{1,2}$

$2P_{0,0} + 8P_{2,2} = 10P_{1,1}$

$2P_{1,2} + 8P_{2,1} = 8P_{2,2}$

$2P_{1,1} = 8P_{2,1}$

The solution to these equations is:

$(P_{0,0}; P_{1,2}; P_{1,1}; P_{2,2}; P_{2,1}) = (64/114, 16/114, 20/114, 9/114, 5/114)$

Hence $P_0 = \frac{64}{114} = .561$

$P_1 = \frac{16+20}{114} = .316$

$P_2 = \frac{9+5}{114} = .123$

$L = \frac{18+14}{52} = .561$

41. (continued)

If the service time is exponential, then the system is an M/M/1 queue limited to K=2 and with $\lambda=2$ and $\mu=4$. So,

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{(1/2)}{(1-1/8)} = \frac{4}{7} = .571$$

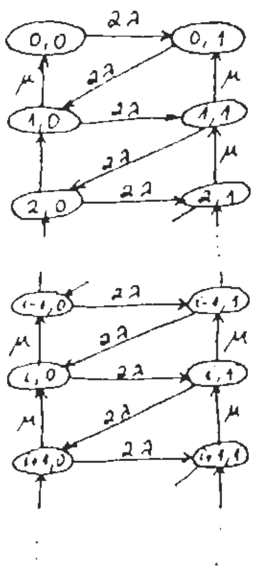
$$P_1 = \left(\tfrac{1}{2}\right) P_0 = \frac{2}{7} = .286$$

$$P_2 = \left(\tfrac{1}{2}\right)^2 P_0 = \frac{1}{7} = .143$$

$$L = \frac{2+2}{7} = \frac{4}{7} = .571$$

42. (a) state is (n, i) where n is the number of customers in the system $(n \geq 1)$ and i is the number of completed arrival stages for currently arriving customer ($i = 0,1$).

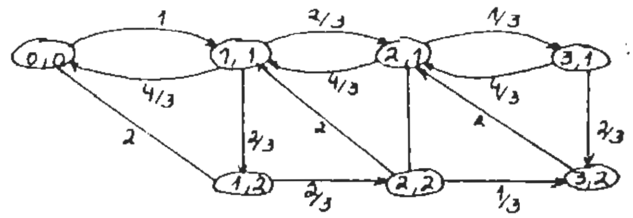
(b)



43. (a) state is (n,i) where n is the number of failed machines ($n=0,1,2,3$) and i is the stage of service (which operation) for the machine under repair ($i = 0$ if no machines are failed), 1, 2)

43. (continued)

(b)



(c)

State	Rate in = Rate out
(0,0)	$4/3\ P_{1,1} + 2\ P_{1,2} = P_{0,0}$
(1,1)	$P_{0,0} + 4/3\ P_{2,1} + 2\ P_{2,2} = (4/3 + 2/3 + 2/3)\,P_{1,1}$
(2,1)	$2/3\,P_{1,1} + 4/3\,P_{3,1} + 2\,P_{3,2} = (4/3 + 2/3 + 1/3)\,P_{2,1}$
(3,1)	$1/3\ P_{2,1} = (4/3 + 2/3)\,P_{3,1}$
(1,2)	$2/3\ P_{1,1} = (2 + 2/3)\,P_{1,2}$
(2,2)	$2/3\,(P_{1,2} + P_{2,1}) = (2 + 1/3)\,P_{2,2}$
(3,2)	$1/3\,P_{2,2} + 2/3\ P_{3,1} = 2\ P_{3,2}$

44. (a) Let T be the repair time.

$E(T) = E(T \mid \text{minor repair needed}).(0.9) +$

$+ E(T \mid \text{major repair needed}).(0.1) = \frac{1}{2}(0.9) + 5(0.1) =$

$= .95$ hours

Now let X be a binary random variable with $P(X=1)=p=0.9$ and $P(X=0)=q=0.1$, Y_i be an exponential random variable with mean $1/\lambda_i$ $(i=1,2)$, with $\frac{1}{\lambda_1}=\frac{1}{2}$ and $\frac{1}{\lambda_2}=5$. Then we may express T as follows:

$T = Y_1.X + Y_2(1-X)$ where X, Y_1, Y_2 are independent

To calculate $\sigma^2 = Var(T)$ we use the formula:

$Var(T) = E(Var(T|X)) + Var(E(T|X))$

$Var(T|X) = Var(Y_1).X + Var(Y_2)(1-X) = (1/\lambda_1^2)X + (1/\lambda_2^2)(1-X)$

$\therefore\ E(Var(T|X)) = p/\lambda_1^2 + q/\lambda_2^2$

$E(T|X) = E(Y_1).X + E(Y_2).(1-X) = \frac{1}{\lambda_1}.X + \frac{1}{\lambda_2}.(1-X) =$

$= \frac{1}{\lambda_2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)X$

$\therefore\ Var(E(T|X)) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2.Var\,X = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq$

Therefore,

$Var(T) = \frac{p}{\lambda_1^2} + \frac{q}{\lambda_2^2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq = 4.5475$

44. (a) (continued)

Now we can see that T has a variance much bigger than that of an exponential random variable with same mean, which would be $(.95)^2 = .9025$

(b) $\left.\begin{array}{l}\mu = \frac{1}{.95} \\ \lambda = 1\end{array}\right\} \Rightarrow \rho = .95$

Since this is an M/G/1 queue we can apply the following formulas:

$P_0 = 1 - \rho = 1. - .95 = .05$

$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = \frac{(4.5475)^2 + (.95)^2}{2 \times .05} = 215.82$

$L = \rho + L_q = 216.77$

$W_q = \frac{L_q}{\lambda} = 215.82$

$W = W_q + \frac{1}{\mu} = 216.77$

(c) $W \mid$ major repair needed $= W_q + 5 = 220.82$

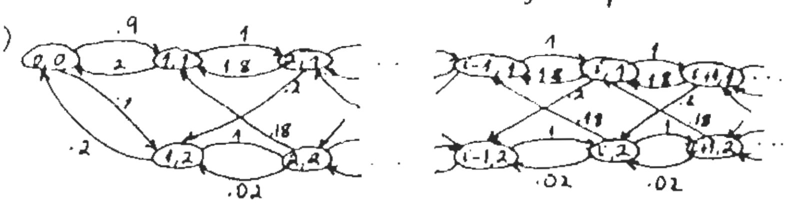
$W \mid$ minor repair needed $= W_q + .5 = 216.32$

$L_{\text{major repair machines}} = (\lambda)(0.1)(220.82) = 22.082$

$L_{\text{minor repair machines}} = (\lambda)(0.9)(216.32) = 194.69$

(d) state is (n,i) where n is the number of failed machines and i is the type of repair being done on machine under repair ($i=1$ denotes minor repair and $i=2$ denotes major repair).

(e)



45. (a) $P_{n_1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}$

$P_{n_2} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_2}$

$P\{(N_1, N_2) = (n_1, n_2)\} = P_{n_1} P_{n_2} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{2}{3}\right)^{n_2}$

(b) $P\{(N_1, N_2) = (0,0)\} = \frac{1}{6}$

(c) $L = L_1 + L_2 = 1 + 2 = 3$

$W = W_1 + W_2 = \frac{1}{10} + \frac{2}{10} = .3$ hour $= 18$ minutes

46.

Facility j	a_j	$j=1$	$j=2$	$j=3$	$\cdots$	$j=m-1$	$j=m$
					P_{ij}		
$j=1$	λ	0	1	0	$\cdots$	0	0
$j=2$	0	0	0	1	$\cdots$	0	0
$\vdots$			$\vdots$	$\vdots$	$\cdots$		$\vdots$
$j=m-1$	0	0	0	0	$\cdots$	0	1
$j=m$	0	0	0	0	$\cdots$	0	0

47. (a) $\lambda_1 = 10 \qquad + .3\lambda_2 + .4\lambda_3$

$\lambda_2 = 15 + .5\lambda_1 \qquad + .5\lambda_3$

$\lambda_3 = 3 + .3\lambda_1 + .2\lambda_2$

$\Rightarrow \lambda_1 = 30,\ \lambda_2 = 40,$ and $\lambda_3 = 20$

(b) $P_{n_1} = (\frac{1}{4})(\frac{3}{4})^{n_1}$

$P_{n_2} = (\frac{1}{5})(\frac{4}{5})^{n_2}$

$P_{n_3} = (\frac{1}{3})(\frac{2}{3})^{n_3}$

$P\{(N_1, N_2, N_3) = (n_1, n_2, n_3)\} = (\frac{1}{60})(\frac{3}{4})^{n_1}(\frac{4}{5})^{n_2}(\frac{2}{3})^{n_3}$

(c) $P\{(N_1, N_2, N_3) = (0, 0, 0)\} = \frac{1}{60}$

(d) $L = L_1 + L_2 + L_3 = 3 + 4 + 2 = 9$

(e) $W = \dfrac{L}{a_1 + a_2 + a_3} = \dfrac{9}{28}$

1.

	Service Costs	Waiting Costs
(a)	Salaries of checkers, cost of cash registers	Lost profit from lost business
(b)	Salaries of firemen, cost of fire trucks	Expected $ destruction due to waiting
(c)	Salaries of toll-takers, cost of constructing toll lane	Social cost of waiting by commuters
(d)	Salaries of repairpersons, cost of tools	Lost profit from lost business
(e)	Salaries of Longshoremen, cost of equipment	Lost profit from ships not loaded or unloaded
(f)	Salary of an operator'as a function of their experience	Lost profit from lost productivity of unused machines
(g)	Salaries of operators, cost of equipment	Lost profit due to lost productivity of materials not handled
(h)	Salaries of plumbers, cost of tools	Lost profit from lost business
(i)	Salaries of employees, cost of equipment	Lost profit from lost business
(j)	Salaries of typists, cost of typewriters	Lost profit due to typing jobs being unfinished

2. $s = 1, \lambda = 2, \mu = 4 \Rightarrow \rho = 1/2 \Rightarrow P_n = (1/2)^{n+1}$ and $f_n(t) = 2e^{-2t}$

The answers to (a) through (d) are based on the following identities:

(i) $$\sum_{n=0}^{\infty} n\,x^n = \frac{x}{(1-x)^2} \qquad \text{if } |x|<1$$

(ii) $$\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \qquad \text{if } |x|<1$$

(iii) $$\sum_{n=0}^{\infty} n^3 x^n = \frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \qquad \text{if } |x|<1$$

(iv) $$\int_0^b x\,e^{-\alpha x}\,dx = \frac{1}{\alpha^2}\left(1 - e^{-\alpha b} - \alpha b\,e^{-\alpha b}\right)$$

(v) $$\int_0^\infty x\,e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

(vi) $$\int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{b^2}{\alpha} e^{-\alpha b} + \frac{2b}{\alpha^2} e^{-\alpha b} + \frac{2}{\alpha^3} e^{-\alpha b}$$

(vii) $$\int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{6}{\alpha^4}$$

(a) $$E[WC] = \sum_{n=0}^{\infty} (10n + 2n^2) P_n = 10 \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n+1} + 2 \sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^{n+1}$$

$$= 5 \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n} + \sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^{n}$$

$$= 5\left(\frac{\frac{1}{2}}{(1-\frac{1}{2})^2}\right) + \left(\frac{2\left(\frac{1}{2}\right)^2}{(1-\frac{1}{2})^3} + \frac{\frac{1}{2}}{(1-\frac{1}{2})^2}\right) \qquad \text{by (i) and (ii)}$$

$$= 16$$

2 (continued)

(b) $$E[WC] = \sum_{n=0}^{2} 10\, n \left(\frac{1}{2}\right)^{n+1} + \sum_{n=3}^{5} 6\, n^2 \left(\frac{1}{2}\right)^{n+1} + \sum_{n=6}^{\infty} n^3 \left(\frac{1}{2}\right)^{n+1}$$

$$= 10\left(\frac{1}{4}\right) + 20\left(\frac{1}{8}\right) + 54\left(\frac{1}{16}\right) + 96\left(\frac{1}{32}\right) + 150\left(\frac{1}{64}\right) + \sum_{n=6}^{\infty} n^3 \left(\frac{1}{2}\right)^{n+1}$$

$$= 20 + \frac{419}{128} = 23.273$$

(c) $$E[WC] = \lambda E[h(w)] = 2\int_0^{\infty} (25\,w + w^3)(2\,e^{-2w})\,dw$$

$$= 100\int_0^{\infty} w\,e^{-2w}\,dw + 4\int_0^{\infty} w^3 e^{-2w}\,dw \quad = \text{by (v) and (vii)}$$

$$= 100 \cdot \frac{1}{2^2} + 4 \cdot \frac{6}{2^4} = 26.5$$

(d) $$E[WC] = 2\int_0^{1} w\,(2\,e^{-2w}\,dw + 2\int_1^{\infty} w^2 (2\,e^{-2w})\,dw$$

$$= 4\left[\frac{1}{2^2}(1 - e^{-2} - 2\,e^{-2})\right] + 4\left[\frac{e^{-2}}{2} + \frac{2\,e^{-2}}{2^2} + \frac{2\,e^{-2}}{2^3}\right]$$

$$= 1 - 3e^{-2} + 5\,e^{-2} = 1 + 2\,e^{-2} = 1.271$$

3. $\lambda = 4$, $\mu = 5$, $c_s = 20$ and $g(n) = \begin{cases} 0 & \text{for } N = 0 \\ 120 & \text{for } N = 1 \\ 120 + 180\,(N-1) & \text{for } N \geq 2 \end{cases}$

$$E[WC] = \sum_{n=0}^{\infty} g(n)\,P_n = 120\sum_{n=1}^{\infty} P_n + 180\sum_{n=2}^{\infty} N \cdot P_n - 180\sum_{n=2}^{\infty} P_n$$

$$= 120(1 - P_0) + 180[L - P_1] - 180[1 - P_0 - P_1] = 60\,P_0 + 180\,L - 60$$

s	$\rho = 4/55$	P_0	L	E[WC]	E[SC]	E[TC]
1	.8	.2	4.0	672.00	20.0	692.00
2	.4	.43	.95	136.80	40.0	176.80
3	.267	.45	.82	114.60	60.0	174.60
4	.2	.44	.80	110.40	80.0	190.40

Hence, $s^* = 3$ and $E[TC] = \$\,174.60$ per hour.

4. (a) Model 2, with s fixed (equal to 1), $A = \{40, 60\}$,

$$f(\mu) = \begin{cases} 10 & \text{for } \mu = 40 \\ 15 & \text{for } \mu = 60 \end{cases}, \lambda = 30$$

That is, we must choose between a slow server (cashier) and a fast server (cashier + box boy)

(b) $$E[WC] = \lambda E[h(W)] = \lambda E[(.05)\,W] = \lambda(.05)\,W = (.05)\,L = (.05) \cdot \frac{\lambda}{\mu - \lambda}$$

4 (a) (continued)

Therefore we have

μ	$sf(\mu)$	$E[WC]$	$E[TC]$	
40	10	.15	10.15	← minimum
60	15	.05	15.05	

Hence the status quo should be maintained.

5. $\lambda=4, \mu_A=5, \mu_B=6\,2/3,\ h(W) = \begin{cases} 20W & \text{for } 0 \le W \le 1 \\ 20W+(W-1)100 & \text{for } W \ge 1 \end{cases}$

$$E[h(W)] = \int_0^1 (20w)(\alpha e^{-\alpha w})\,dw + \int_1^\infty [20w + 100(w-1)]\,\alpha e^{-\alpha w}\,dw$$

where $\alpha = \mu(1-\rho)$. Simplifying, $E[h(W)] = \dfrac{20}{\alpha}(1+4e^{-\alpha})$ where $\alpha = \begin{cases} 1 & \text{for A} \\ 8/3 & \text{for B} \end{cases}$

So $E[WC] = \lambda E[h(W)] = \begin{cases} 80(1+4e^{-1}) & \text{for A} \\ 30(1+4e^{-8/3}) & \text{for B} \end{cases}$

Alternative	E(SC)	E[WC]	E[TC]
A	50	197.72	247.72
B	150	38.34	188.34

Hence, purchase equipment B, E [TC] = \$188.34 per hour.

6. For alternative 1, the system has Poisson input with $\lambda = 1/5$, constant service with $\mu = 1/6$ and $s = 2$. For Alternative 2, the system has Poisson input with $\lambda = 1/5$, constant service with $\mu = 1/3$ and $s = 1$.

Alternative	E[SC]	ρ	L (from Fig. 16.11)	E[WC] = 50 L	E[TC]
1	34.25	3/5	1.2	60	94.25
2	45.66	3/5	1.0	50	95.66

Hence, Alternative 1 should be chosen, E [TC] = \$94.25 per hour.

7. For alternative A, the system has Poisson input with $\lambda = 1/45$, Erlang service with $k = 2$ and $\mu = 1/36$ and $s = 2$. For Alternative B, the system has Poisson input with $\lambda = 1/45$, Erlang service with $k = 2$ and $\mu = 1/18$ and $s = 1$. Planes arrive at the rate of one every 45 hours or, equivalently, $8760/45 \approx 195$ arrivals per year. Since each plane goes to the maintenance shop five times a year, there are approximately $195/5 = 39$ planes.
So, for Alternative A, E [SC] = \$100,000 x 39 / 8760 = \$1780.82 per hour
and for Alternative B, E [SC] = \$550,000 x 39 / 8760 = \$2448.63 per hour.

Also for Alternative A, $\rho = \dfrac{\left(\frac{1}{45}\right)}{(2)\left(\frac{1}{36}\right)} = \dfrac{2}{5} \Rightarrow L = .91$ from Fig. 16.13

and for Alternative B, $L = \left(\dfrac{1+k}{2k}\right)\left(\dfrac{\lambda^2}{\mu(\mu-\lambda)}\right) + \dfrac{\lambda}{\mu} = \dfrac{3}{4}\left[\dfrac{\left(\frac{1}{45}\right)^2}{\left(\frac{1}{18}\right)\left(\frac{1}{18}-\frac{1}{45}\right)}\right] + \dfrac{\left(\frac{1}{45}\right)}{\left(\frac{1}{18}\right)} = \dfrac{3}{5}$

Alternative	E[SC]	E[WC] = 3000L	E[TC]
A	1780.82	2730.00	4510.82
B	2448.63	1800.00	4248.63

So choose Alternative B with E [TC] = \$4248.63 per hour.

8. For Status Quo, the system has Poisson input with $\lambda = 3$, expenential service with $\mu = 4$ and $s = 1$. So $L = \frac{\lambda}{\lambda - \mu} = 3$.

For the Proposal, the system has Poisson input with $\lambda = 3$, and general service with $\mu = \frac{1}{\left(\frac{1}{10} + \frac{1}{5}\right)} = \frac{10}{3}$

and

$\sigma^2 = \frac{1}{200} + \frac{1}{50} = \frac{1}{40}$ with $s = 1$. So,

$$L = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{\left(\frac{9}{40}\right) + \left(\frac{9}{10}\right)^2}{2\left(1 - \frac{9}{10}\right)} + \frac{9}{10} = 6.075$$

Alternative	E[SC]	E[WC] = 10 L	E[TC]
Status Quo	20	30.00	50.00
Proposal	15	60.75	75.75

So continue Status Quo with E [TC] = \$50.00 per hour.

9. For Alternative 1, the system has Poisson input with $\lambda = 0.3$, exponential service with $\mu = 0.2$, and $s = 2$. So

$$P_0 = \left[1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 \cdot \frac{1}{1 - \frac{\lambda}{2\mu}}\right]^{-1} = \left[1 + \frac{3}{2} + \frac{9}{8} \cdot \frac{1}{1 - \frac{3}{4}}\right]^{-1} = \frac{1}{7}$$

$$\therefore L = \frac{P_0 \left(\frac{\lambda}{\mu}\right)^2 \rho}{2!(1-\rho)^2} + \frac{\lambda}{\mu} = \frac{\frac{1}{7} \cdot \frac{9}{4} \cdot \frac{3}{4}}{2 \cdot \left(\frac{1}{4}\right)^2} + \frac{3}{2} = \frac{24}{7} = 3.43.$$

For alternative 2, the system has Poisson input with $\lambda = 0.3$, general service (sum of E (μ_1, k_1) and E (μ_2, k_2) where $k_1 = k_2 = 4$, $\mu_1 = 0.5$ and $\mu_2 = 1$ - E (μ, k) denotes Erlang distribution with parameters μ and k, see Section 16.7) with $\mu = 1/3$ and $\sigma^2 = 5/4$

$$\text{So } L = \rho + \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)} = 0.9 + \frac{0.9 \cdot \frac{5}{4} + 0.81}{0.2} = 5.51$$

Alternative	E[SC]	E[WC] = 100 L	E[TC]
1	2000	343	2343
2	1500	551	2051

So choose Alternative 2, with E [TC] = 2051 per week.

10. For the status quo, the system has Poisson input with $\lambda = 15$, exponential service with $\mu = 15$, $s = 1$ and limited waiting room with $k = 4$. There is a waiting cost of 6 W_q for each customer due to loss of good will and also a waiting cost of \$45 per hour when the system is full (4 cars in system) due to loss of potential customers. So

$E[TC] = E[WC] = \lambda 6 W_q + 45 P_4 = 6 L_q + 45 P_4$.

In this case, $\frac{\lambda}{\mu} = 1$, so

$$P_0 = \frac{1}{k+1}, \quad P_n = \frac{1}{k+1} \quad n = 1, 2, \ldots, k \quad \therefore P_4 = \frac{1}{5}$$

$$L = \sum_{n=1}^{k} n P_n = \frac{1}{k+1} \sum_{n=1}^{k} n = \frac{k(k+1)}{2(k+1)} = \frac{k}{2}$$

$$L_q = L - (1 - P_0) = \frac{k}{2} - 1 + \frac{1}{k+1} = \frac{4}{2} - 1 + \frac{1}{5} = \frac{6}{5}$$

Thus, E [TC] = 6 (6 / 5) + 45 (1 / 5) = \$16.20 per hour.

For Proposal 1, the system has Poisson input with $\lambda = 15$, exponential service with $\mu = 20$ and $s = 1$. In addition to the waiting cost of 6 L_q due to los of good will, there is an expected waiting cost of \$2 per

10. (continued)

customer that waits longer than 1/2 hour before his car is ready. The expected value of this additional waiting cost is given by

$$2\lambda P\{W > 1/2\} = 2\lambda e^{-\mu(1-\rho)/2} = 30e^{-2.5} = 2.46.$$

Also, $6L_q = \dfrac{6\lambda^2}{\mu(\mu-\lambda)} = \dfrac{6 \times 225}{20 \times 5} = 13.50$

Hence, E[TC] = 3 + 13.50 + 2.46 = 18.96 per hour (where 3 is the capitalized cost of the new equipment.)

For Proposal 2, the system has a Poisson input with $\lambda = 15$, Erlang service with $\mu = 30$ and k = 2, and s = 1. The only waiting cost is $6L_q$ due to loss of good will. In this case,

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) = \left(\frac{3}{4}\right)\left(\frac{225}{30 \times 15}\right) = \frac{3}{8} = 0.375.$$ Hence, E[TC] = 10 + 2.25 = 12.25.

Summarizing, Status Quo: $E[TC] = 0 + 6L_q + 45P_4 = 16.20$

Proposal 1: $E[TC] = 3 + 6L_q + 30P\{W \geq 1/2\} = 18.96$

Proposal 2: $E[TC] = 10 + 6Lq = 10 + 2.25 = 12.25$.

Hence, Proposal 2 should be adopted.

11. (a)

s	$L = \frac{\lambda}{\mu_s - \lambda}$	E[WC] = 15L	E[SC] = 10s	E[TC]
1	∞	∞	10	∞
2	1	15.00	20	35.00
3	1/2	7.50	30	37.50

Since clearly E[SC] > 35 for $s \geq 4$, it follows that $s^* = 2$

(b)

s	$\mu_s = \sqrt{s}$	$L = \frac{\lambda}{\mu_s - \lambda}$	E[WC] = 15L	E[SC] = 10s	E[TC]
1	1.0	∞	∞	10	∞
2	1.414	2.414	36.21	20	56.21
3	1.732	1.366	20.49	30	50.49
4	2.0	1.000	15.00	40	55.00
5	2.236	.809	13.75	50	63.75

Since clearly E[SC] > 50.49 for $s \geq 6$, it follows that $s^* = 3$.

12. This is an $M/E_2/1$ model with $\lambda = 3$ and so:

$$E[TC] = 40{,}000 + 10{,}000\mu + 90{,}000\left[\frac{27}{4\mu(\mu-3)} + \frac{3}{\mu}\right]$$

$$\frac{dE[TC]}{d\mu} = 10{,}000 - 90{,}000\left[\frac{27}{4\mu(\mu-3)^2} + \frac{27}{4\mu^2(\mu-3)} + \frac{3}{\mu^2}\right] = 0.$$ Solving for μ we get $\mu = 7.774$,

at which point $E[TC] = \$168{,}840/wk$.

If we check $$\frac{d^2E[TC]}{d\mu^2} = 90{,}000\left[\frac{6}{\mu^3} + \frac{27}{2\mu(\mu-3)^3} + \frac{27}{2\mu^2(\mu-3)^2} + \frac{27}{2\mu^3(\mu-3)}\right] \geq 0$$ if $\mu \geq 3$.

13. $E[SC] = .10\mu^2$ and $E[WC] = .20L = \frac{.20(1/2)}{\mu - 1/2} = \frac{.10}{\mu - 1/2}$.

So $E[TC] = .10\mu^2 + \frac{.10}{\mu - 1/2}$

μ	$E[TC]$
.5	∞
.75	.45625
1.00	.30000
1.10	.28767
1.15	.28610
1.20	.28656
1.25	.28958
1.50	.32500
1.75	.38625
2.00	.46667

So $\mu^* = 1.15$

14. Given that $s=1$ (by the Optimality of the Single Server result), $E[TC] = C_r\mu + C_w L = C_r\mu + C_w\left(\frac{\lambda}{\mu-\lambda}\right)$

$$\frac{dE[TC]}{d\mu} = C_r - \frac{C_w\lambda}{(\mu-\lambda)^2} = 0 \Rightarrow (\mu-\lambda)^2 = \frac{\lambda C_w}{C_r}$$

$$\Rightarrow \mu = \lambda + \sqrt{\lambda C_w / C_r}$$

$$\frac{d^2E[TC]}{d\mu^2} = \frac{2\lambda C_w}{(\mu-\lambda)^3} > 0 \text{ for all } \mu > \lambda$$

Hence, $\mu = \lambda + \sqrt{\lambda C_w/C_r}$ is the minimizing value.

15. $E[TC] = D\mu + \lambda C/(\mu-\lambda)^2$

$$\frac{dE[TC]}{d\mu} = D - \frac{2\lambda C}{(\mu-\lambda)^3} = 0 \Rightarrow (\mu-\lambda)^3 = \frac{2\lambda C}{D} \Rightarrow \mu = \lambda + \left(\frac{2\lambda C}{D}\right)^{1/3}$$

Also, $\frac{d^2E[TC]}{d\mu^2} = \frac{6\lambda C}{(\mu-\lambda)^4} > 0$ when $C > 0$

Hence, $\mu = \lambda + \left(\frac{2\lambda C}{D}\right)^{1/3}$ is the minimizing value.

16. (a) Part (a) of Problem 21 is a special case of Model 3, in which we want to determine λ (since determining the number of machines assigned to one operator is equivalent to determining λ, the mean arrival rate) and s is fixed (equal to 1 in this case).

(b) (i) the resulting system would be a M/M/s with finite calling population (size of population is equal to the total number of machines) and the associated decision problem fits Model 1 (unknown s).

16 (b). (Continued)

(ii) the resulting system is a collection of independent M/M/1 with finite calling populations and the appropriate decision model is a combination of Models 2 and 3 since in this case we have to determine μ (the service rate of a crew depends on how many operators are assigned to the crew) and λ (the mean arrival rate depends on the number of machines assigned to a crew) and s is fixed equal to 1.

(iii) the resulting system doesn't fit any of the models described in Chapter 16.

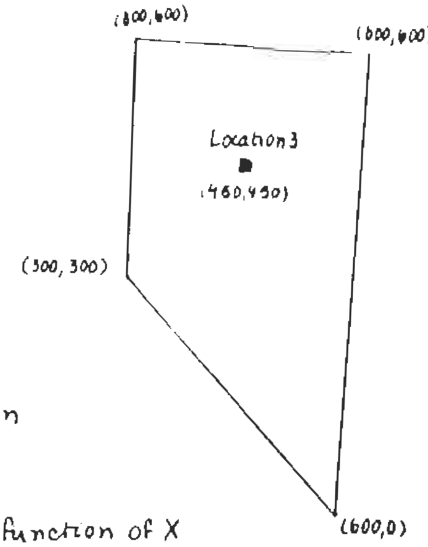
Each of the proposed alternatives allows, in varied degrees, the sharing of resources (the operators) in contrast to the original proposal. Now, since in the original proposal the operators would be idle most of the time, it is reasonable to expect that if we allow interaction, the production rate obtained with the same number of operators will increase, and thus, the required number of operators that are needed to achieve the desired production rate will decrease.
What might prevent this from happening?
In alternatives (i) and (iii), the travel time (not taken into account in the above argument) may pose a problem (instead of turning idle time into service time we may be turning idle time into travel time). Also, in alternative (iii) the service rate of a group of workers may be smaller than the individual service rate, since they won't be working together regularly. And finally, although this is not the case in Alternative (ii) (the components of a crew do work together regularly) even then we may have that $\mu' < n\mu$ (where μ' is the service rate of crew of n operators and μ is the individual service rate).

17. $a = b = c = d = 300$ and $v = 3$ miles/hour $= 264$ feet/min

So $E[T] = \dfrac{1}{264}\left[\dfrac{(300)^2+(300)^2}{(300+300)} + \dfrac{(300)^2+(300)^2}{(300+300)}\right] =$

$= 600/264$ minutes $= 2.27$ minutes

18. - Area assigned to tool Crib in Location 3, in Alternative 3



- The first step is to relabel Location 3 as the origin (0,0) for an (x,y) coordinate system (subtract 450 from all coordinates shown in figure on the right)

- The probability density function of X is obtained by using the height of the area assigned to the tool crib at Location 3 for each possible value of X = x and then dividing by the size of the area, as given in figure 1.(a) below This then yields the uniform distribution of |X| shown in figure 1 (b).

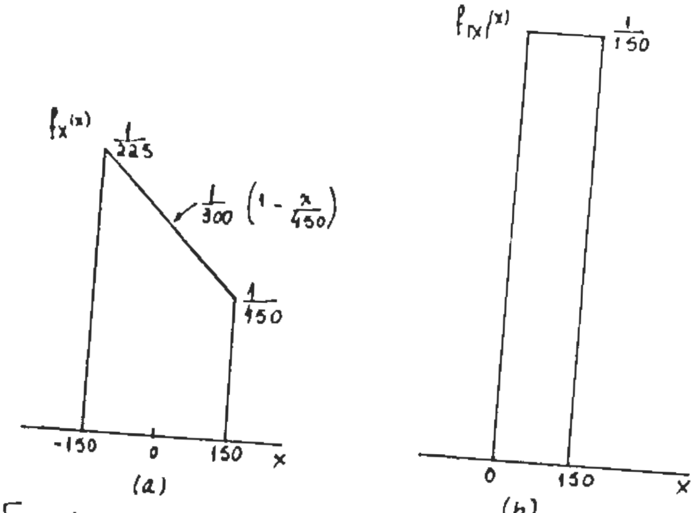


Fig. 1 Probability density functions of (a) X, and (b) |Y|

Thus, $E\{|X|\} = \dfrac{1}{150}\displaystyle\int_0^{100} x\,dx = 75$

18. (Continued)

- The probability density function of Y is obtained by using the width of the area assigned to tool crib at Location 3 for each possible value of Y = y and then dividing by the size of the area, as given in figure 2.(a). This then leads to the probability density function of |Y| shown in figure 2.(b).

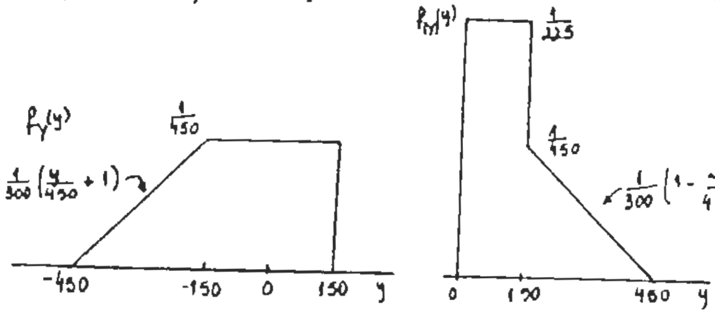


Fig 2 Probability density functions of (a) Y, and (b) |Y|.

Thus,

$$E\{|Y|\} = \frac{1}{225}\int_0^{150} y\,dy + \frac{1}{300}\int_{150}^{450}\left(1-\frac{y}{450}\right)y\,dy =$$

$$= 133\ 1/3$$

Consequently, E(T) for the tool crib in Location 3 is

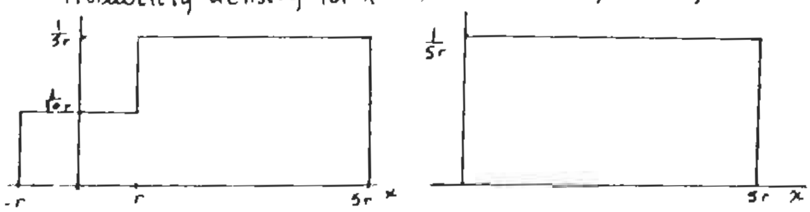
$$E(T) = \frac{2}{v}\left(E\{|X|\} + E\{|Y|\}\right) =$$

$$= \frac{2}{15{,}000}\left(75 + 133\ 1/3\right)$$

$$= 0.0278 \text{ hr}$$

19. (a) Total area $= (2r)^2 + (4r)^2 = 20r^2$

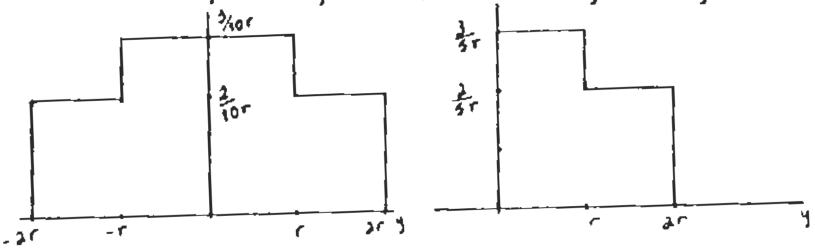
Probability density for X => Probability density for |X|



So, $E\{|X|\} = \int_0^{5r}\left(\frac{1}{5r}\right)x\,dx = 2.5r$

19 (a) (continued)

Probability density for Y $\Rightarrow$ Probability density for |Y|



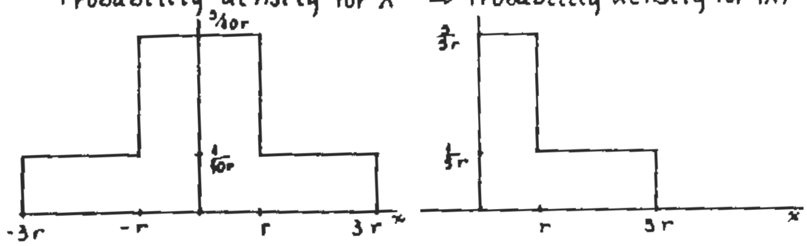
So, $E\{|Y|\} = \int_0^r \left(\frac{3}{5r}\right) y\,dy + \int_r^{2r} \left(\frac{2}{5r}\right) y\,dy = .9r$

Hence, $E[T] = \frac{2}{v}(2.5 + .9)r = \frac{6.8r}{v}$

(b) Area is symmetric about (0,0) so $E\{|X|\} = E\{|Y|\}$

Total area $= 5(2r)^2 = 20r^2$

Probability density for X $\Rightarrow$ Probability density for |X|

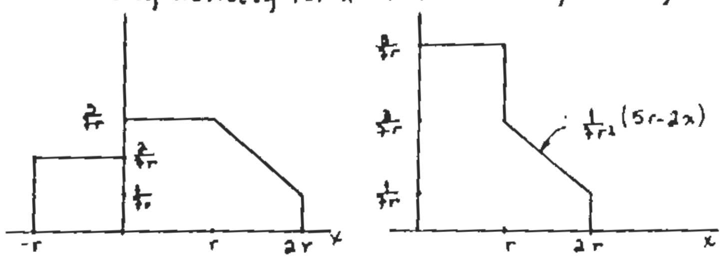


So, $E\{|X|\} = \int_0^r \left(\frac{2}{5r}\right) x\,dx + \int_r^{3r} \left(\frac{1}{5r}\right) x\,dx = 1.1r$

Hence, $E[T] = \frac{2r}{v}(1.1 + 1.1) = \frac{4.4r}{v}$

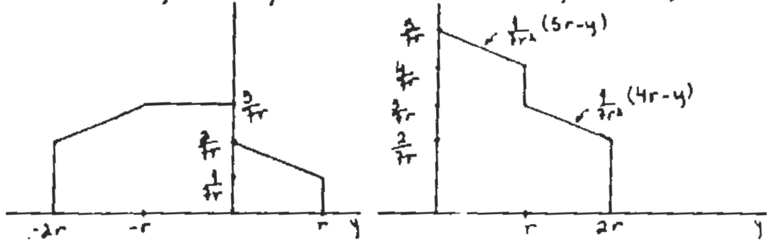
(c) Total area $= 2(2r^2 + r^2 + \tfrac{1}{2}r^2) = 7r^2$

Probability density for X $\Rightarrow$ Probability density for |X|



So, $E\{|X|\} = \int_0^r \left(\frac{5}{7r}\right) x\,dx + \int_r^{2r} \left(\frac{1}{7r^2}\right)(5r-2x)\,x\,dx = \frac{16}{21}r$

Probability density for Y $\Rightarrow$ Probability density for |Y|



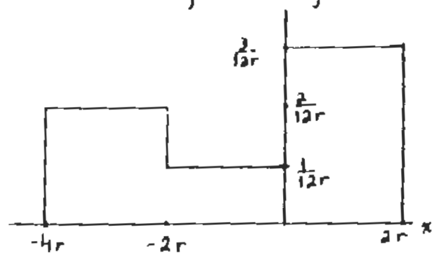
19 (c) (continued)

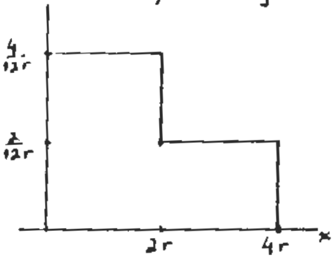
So, $E\{|Y|\} = \frac{1}{7r^2}\left\{\int_0^r (5r-y)y\,dy + \int_r^{2r}(4r-y)y\,dy\right\} = \frac{5r}{6}$

Hence, $E(T) = \frac{2r}{v}\left[\frac{(32+35)}{42}\right] = \frac{3.19\,r}{v}$

(d) Total area $= 6(4r^2) = 24r^2$

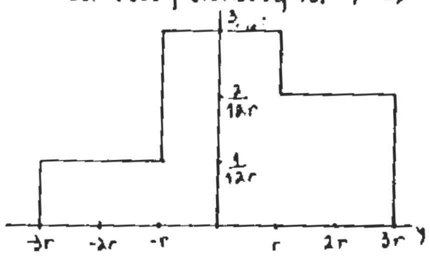
Probability density for $X \Rightarrow$ Probability density for $|X|$

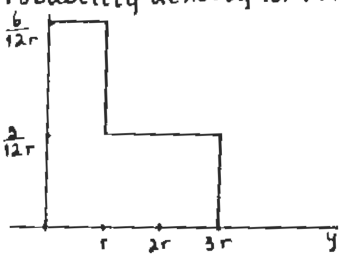




So, $E\{|X|\} = \int_0^{2r}\left(\frac{4}{12r}\right)x\,dx + \int_{2r}^{4r}\left(\frac{2}{12r}\right)x\,dx = \frac{5r}{3}$

Probability density for $Y \Rightarrow$ Probability density for $|Y|$

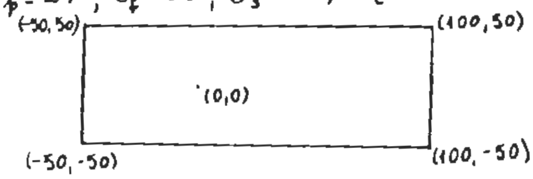




So, $E\{|Y|\} = \int_0^r \left(\frac{6}{12r}\right)y\,dy + \int_r^{3r}\left(\frac{3}{12r}\right)y\,dy = \frac{5r}{4}$

So, $E(T) = \frac{2r}{v}\left(\frac{5}{3}+\frac{5}{4}\right) = \left(\frac{35}{6}\right)\frac{r}{v} = \frac{5.83\,r}{v}$

20 $\mu = 30$, $s = 1$, $\lambda_p = 24$, $C_f = 20$, $C_s = 15$, $C_t = 25$.



$\underline{n=1}$: $\lambda = \frac{\lambda_p}{n} = 24$

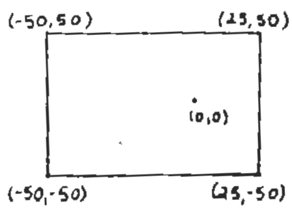
So $a = 50$, $b = 50$, $c = 100$, $d = 50$

$\therefore\ E(T) = \frac{1}{5{,}000}\left(\frac{50^2+100^2}{50+100} + \frac{50^2+50^2}{50+50}\right) = .0267$ hrs.

Also, $L = \frac{\lambda}{\mu-\lambda} = \frac{24}{30-24} = 4$

$\underline{n=2}$: $\lambda = \frac{\lambda_p}{n} = 12$

Relabelling symmetric areas,



So $a = 50$, $b = 50$, $c = 25$, $d = 50$

20. (continued)

$$E(T) = \frac{1}{5000}\left(\frac{50^2+25^2}{50+25} + \frac{50^2+50^2}{50+50}\right) = .0183 \text{ hrs.}$$

$$\text{Also, } L = \frac{\lambda}{\mu-\lambda} = \frac{12}{30-12} = \frac{2}{3}$$

$$\text{Hence, } E(TC) = n\left[(C_f + C_s) + C_t L + \frac{24}{n} C_t E(T)\right]$$

n	λ	$E[T]$	L	C_f+C_s	$C_t L$	$\lambda C_t E(T)$	$E[TC]$
1	24	.0267	4	35	100	16	151
2	12	.0183	2/3	35	50/3	5.5	114.33

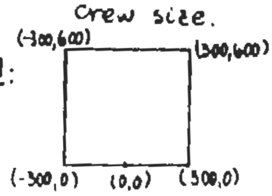
So there should be two facilities.

21. Given: $C_f = 10$, $C_m = 15$, $C_t = 40$, $\lambda_p = 90$, $v = 20000$ feet/hour

E(loading time) = 1/20 hours

For unloading, $\mu_m = 30m$ where m is the crew size.

<u>$n=1$</u>:

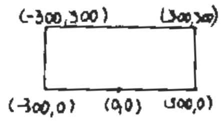


So $a = 300$, $b = 0$, $c = 300$ and $d = 600$

$$\therefore E(T) = \frac{1}{20000}\left[\frac{300^2+300^2}{300+300} + \frac{600^2}{600}\right] = .045 \text{ hrs.}$$

$$\text{Also, } L = \frac{\lambda}{\mu_m - \lambda} = \frac{90}{30m-90} = \frac{3}{m-3}$$

<u>$n=2$</u>: Labelling the two symmetric areas:

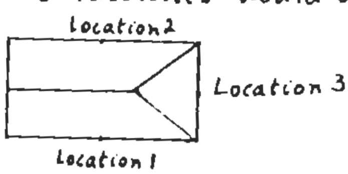


So $a = 300$, $b = 0$, $c = 300$, $d = 300$

$$\text{So } E(T) = \frac{1}{20000}\left[\frac{300^2+300^2}{300+300} + \frac{300^2}{300}\right] = .030 \text{ hrs.}$$

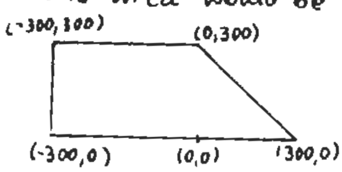
$$\text{Also, } \lambda = \frac{\lambda_p}{n} = 45 \Rightarrow L = \frac{45}{30m-45} = \frac{3}{2m-3}$$

<u>$n=3$</u>: The facilities would be located as follows



<u>Consider Locations 1 and 2</u> that are symmetric

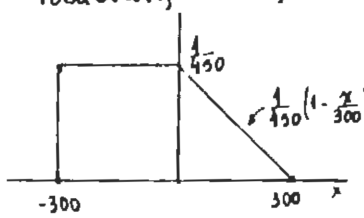
The area would be labelled as

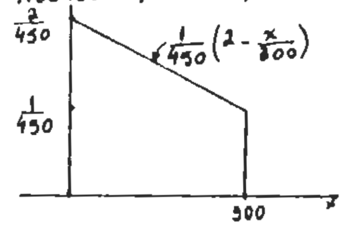


Total area = 135,000

21. (continued)

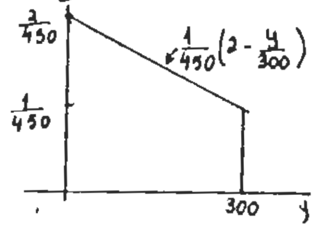
Probability density of X $\Rightarrow$ Probability density of |X|





So $E\{|X|\} = \int_0^{300} \left(\frac{1}{450}\right)\left(2-\frac{x}{300}\right) x \, dx = \frac{400}{3}$

Probability density of $Y = |Y|$:

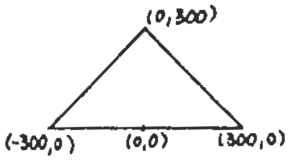


So $E\{|Y|\} = E\{|X|\} = 400/3$

$\therefore E(T) = \frac{2}{20000}\left[\frac{400}{3} + \frac{400}{3}\right] = \frac{4}{150} = .0267$

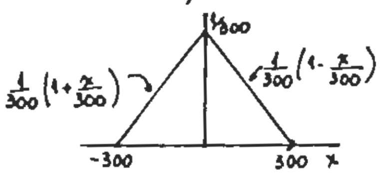
Also, $\lambda = 33\,3/4 \Rightarrow L = \frac{135/4}{30m - 135/4} = \frac{135}{120m - 135} = \frac{9}{8m-9}$

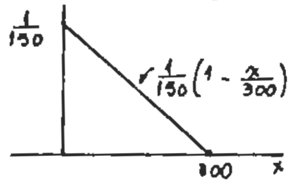
Now consider Location 3 : The area would be labelled



Total area = 90000

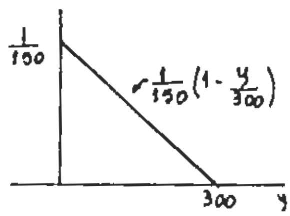
Probability density of X $\Rightarrow$ Probability density of |X|





So $E\{|X|\} = \int_0^{300} \frac{1}{150}\left(1-\frac{x}{300}\right) x \, dx = 100$

Probability density of $Y = |Y|$:



So $E\{|Y|\} = E\{|X|\} = 100$

$\therefore E(T) = \frac{2}{20000}(100 + 100) = \frac{4}{200} = .020$ hrs.

Also, $\lambda = 22\,1/2 \Rightarrow L = \frac{45/2}{30m - 45/2} = \frac{45}{60m-45} = \frac{3}{4m-3}$

21. (continued)

$n=4$: The areas served by the four facilities would be identical to that of Location 3 for $n=3$.

So $E(T) = 4/200 = .020$ hrs. and $L = \frac{3}{4m-3}$

Summarizing:

n	$E(T)$	L
1	.045 hrs.	$3/(m-3)$
2	.030 hrs.	$3/(2m-3)$
3	Locations 1,2; Location 3 .0267 hrs. .020 hrs.	Locations 1,2; Location 3 $9/(8m-9)$ $3/(4m-3)$
4	.020 hrs.	$3/(4m-3)$

$n=1$: $E(TC) = (C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_t/20$ where $\lambda = 90$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
4	3	.045	70	120	162	180	532.00
5	1.5	.045	85	60	162	180	487.00
6	1	.045	100	40	162	180	482.00
7	.75	.045	115	30	162	180	487.00

So for $n=1$, the minimum cost per hour is \$482.00 with $m=6$.

$n=2$: $E(TC) = 2[(C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_t/20]$, where $\lambda = 45$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
2	3	.030	40	120	54	90	608.00
3	1	.030	55	40	54	90	478.00
4	.6	.030	70	24	54	90	476.00
5	3/7	.030	85	17.14	54	90	492.29

So for $n=2$, the minimum cost per hour is \$476.00 with $m=4$.

$n=3$: At Locations 1 and 2 where $\lambda = 33\,3/4$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
2	9/7	.0267	40	51.43	36	67.5	194.93
3	3/5	.0267	55	24	36	67.5	182.50
4	9/23	.0267	70	15.65	36	67.5	189.15

At Location 3, where $\lambda = 22\,1/2$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
1	3	.020	25	120	18	45	208.00
2	3/5	.020	40	24	18	45	127.00
3	1/3	.020	55	13.33	18	45	131.33

So, for $n=3$, the minimum cost per hour is $2(182.50) + 127.00 = 492.00$ with $m=3$ at Locations 1 and 2, and $m=2$ at Location 3.

$n=4$: Since all areas served are symmetric to that of Location 3 for $n=3$, the minimum cost per hour is $4(127.00) = 508.00$ with $m=2$.

21. (continued)

Summarizing,

n	m	E(TC)
1	6	482.00
2	4 at both Locations	476.00
3	3 at Locations 1 and 2 2 at Location 3	492.00
4	2 at all Locations	508.00

Therefore, there should be two facilities with crew size of 4.

22. From Table 16.4 and adjacent text

	s = 1	s = 2
$W_1 - 1/\mu$	.024	.00037
$W_2 - 1/\mu$	.154	.00793
$W_3 - 1/\mu$	1.033	.06542

Note that $\lambda_1 = .2$, $\lambda_2 = .6$ and $\lambda_3 = 1.2$

S	E[waiting costs]				E[SC]	E[TC]
	critical	serious	stable	total		
1	480.00	92.40	12.40	584.80	40.00	624.80
2	7.40	4.76	0.79	12.95	80.00	92.95

So there should be two doctors.

23. The system is a non-preemptive queueing system with $\mu = 4$, $\lambda_1 = 6$, $\lambda_2 = 4$ and $\lambda_3 = 2$, where Class 1 is government jobs, Class 2 is commercial jobs and Class 3 is standard jobs. The problem is to determine whether $s = 4$ or $s = 5$.

For $s = 4$, $A = 31.41$, $B_0 = 1$, $B_1 = 5/8$, $B_2 = 3/8$, $B_3 = 1/4$

$W_1 = .30094$

$W_2 = .38585$

$W_3 = .58962$

For $s = 5$, $A = 52.35$, $B_0 = 1$, $B_1 = 7/10$, $B_2 = 1/2$, $B_3 = 2/5$

$W_1 = .27729$

$W_2 = .30458$

$W_3 = .34552$

S	E[WC]				E[SC]	E[TC]
	Government	Commercial	Standard	Total		
4	6771.15	3472.65	884.43	11128.23	5000	16128.23
5	6239.02	2741.22	518.28	9498.52	6250	15748.52

So two additional lathes should be obtained.